Government of **Western Australia**
School Curriculum and Standards Authority

WACE
Western Australian Certificate of Education

MATHEMATICS: SPECIALIST

3C/3D
Calculator-free

WACE Examination 2012

Marking Key

Marking keys are an explicit statement about what the examiner expects of candidates when they respond to a question. They are essential to fair assessment because their proper construction underpins reliability and validity.

When examiners design an examination, they develop provisional marking keys that can be reviewed at a marking key ratification meeting and modified as necessary in the light of candidate responses.

Section One: Calculator-free **(50 Marks)**

Question 1 **(6 marks)**

(a) Evaluate $\int_{-\pi}^{\pi} |\sin x| \ dx$. (3 marks)

Solution
$\int_{-\pi}^{\pi} \|\sin x\| dx = 2\int_{0}^{\pi} \sin x \, dx$ $= 2\left[-\cos x\right]_0^{\pi}$ $= 2\left[1-(-1)\right]$ $= 4$
Specific behaviours
✓ removes the modulus signs using the symmetry of the integrand ✓ integrates trigonometric function correctly ✓ evaluates and simplifies

(b) Determine $\int 6x^2 e^{x^3+9} dx$. (3 marks)

Solution
If we substitute $u = x^3 + 9$ so that $\frac{du}{dx} = 3x^2$, it follows that $\int 6x^2 e^{x^3+9} \, dx = \int 2e^u \, du = 2e^u + c = 2e^{x^3+9} + c$
Specific behaviours
✓ identifies the appropriate substitution ✓ integrates correctly ✓ includes an arbitrary constant

Question 2 **(8 marks)**

Let the 2×2 matrix $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$.

(a) Evaluate A^2 and A^3. (2 marks)

Solution
$A^2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ $A^3 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$
Specific behaviours
✓ correctly evaluates A^2 ✓ correctly evaluates A^3

(b) Given your answers to (a), conjecture the form of the matrix A^n where n is a positive integer. Use proof by mathematical induction to establish the truth of your conjecture. (4 marks)

Solution
From the form of the answers in (a) it is reasonable to assert that $A^n = \begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix}$. For $n = 1$ we have $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. Assume result true for $n = k$ so that $A^k = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$. Then necessary to prove the result for $n = k+1$; i.e. need to prove $A^{k+1} = \begin{bmatrix} 1 & 0 \\ k+1 & 1 \end{bmatrix}$. Now $A^{k+1} = A^k . A = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ k+1 & 1 \end{bmatrix}$, as required. Shown to be true for $n = 1$ and as the truth for $n = k$ implies the result for $n = k+1$ it follows that the conjecture is true for all positive integers.
Specific behaviours
✓ makes an appropriate conjecture for the form of A^n given the answers to Part (a) ✓ states that the conjecture holds for $n = 1$ and writes statement for $n = k$. ✓ proves case for $n = k+1$ ✓ deduces the truth of the results for $n = 1, 2, 3.....$ etc

(c) Determine the effect of applying the matrix A^2 to the rectangle with vertices (1,0), (3,0), (3,1) and (1,1). What type of transformation does the matrix A^2 represent? (2 marks)

Solution
As $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 3 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 3 & 1 \\ 2 & 6 & 7 & 3 \end{bmatrix}$, so the new vertices are (1, 2), (3, 6), (3, 7) and (1, 3). These are the vertices of a parallelogram. It follows that the matrix operates as a shear of factor 2 parallel to the y-axis.
Specific behaviours
✓ correctly determines the image vertices (or states it is a parallelogram) ✓ recognises the transformation as a shear with direction and scale stated correctly

Question 3 **(7 marks)**

(a) Evaluate $\int_0^{\frac{\pi}{3}} \frac{6\sin 3x - 8\cos 2x}{\cos 3x + 2\sin 2x}\, dx$. (3 marks)

Solution
If we substitute $u = \cos 3x + 2\sin 2x$ then the integral becomes $\int_0^{\frac{\pi}{3}} \frac{6\sin 3x - 8\cos 2x}{\cos 3x + 2\sin 2x} dx = \int_{x=0}^{x=\frac{\pi}{3}} -\frac{2}{u}\frac{du}{dx} dx$ $= \int_1^{-1+\sqrt{3}} \frac{-2}{u}\, du$ $= \left[-2\ln u\right]_1^{-1+\sqrt{3}}$ $= -2\ln\left(-1+\sqrt{3}\right)$
Specific behaviours
✓ uses an appropriates substitution ✓ changes the limits ✓ follows through correctly with the integration OR ✓ recognises $\int \frac{1}{u}\, du$ integral ✓ substitutes the limits correctly ✓ integrates correctly

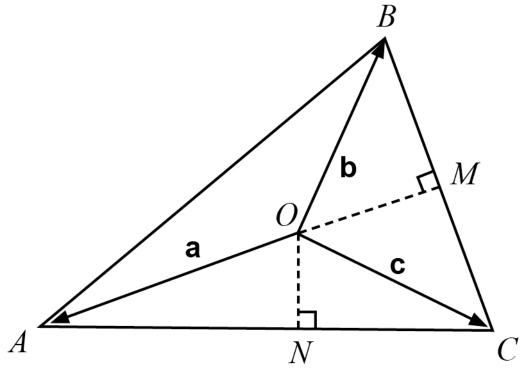
(b) Use the substitution $u = \ln x$ to determine $\int \frac{\sqrt{\ln x} + \ln\sqrt{x}}{x}\, dx$. (4 marks)

Solution
If $u = \ln x$ then $\frac{du}{dx} = \frac{1}{x}$ and $\int \frac{\sqrt{\ln x} + \ln\sqrt{x}}{x} dx = \int \left(\sqrt{u} + \frac{1}{2}u\right)\frac{du}{dx}\, dx$ $= \int \left(u^{\frac{1}{2}} + \frac{u}{2}\right) du$ $= \frac{2}{3}u^{\frac{3}{2}} + \frac{1}{4}u^2 + c$ $= \frac{2}{3}(\ln x)^{\frac{3}{2}} + \frac{1}{4}(\ln x)^2 + c$
Specific behaviours
✓ recognises that $\ln\sqrt{x} = \frac{1}{2}u$ ✓ accurately substitutes for u into integrand ✓ integrates correctly ✓ expresses answer in terms of x

Question 4 **(6 marks)**

In the triangle ABC the sides BC and AC have midpoints M and N respectively. Lines perpendicular to these sides are drawn through their respective midpoints and intersect at the point O.

Denote the line segments $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{c}$.



(a) Express $\overrightarrow{ON}$ in terms of **a** and **c** and $\overrightarrow{OM}$ in terms of **b** and **c**. (1 mark)

Solution
$\overrightarrow{ON} = \frac{1}{2}(\mathbf{c} + \mathbf{a})$ Similarly $\overrightarrow{OM} = \frac{1}{2}(\mathbf{c} + \mathbf{b})$,
Specific behaviours
✓ states $\overrightarrow{ON}$ and $\overrightarrow{OM}$ correctly

(b) Show that $|\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}|$. (2 marks)

Solution
Since $\overrightarrow{OM}$ and $\overrightarrow{ON}$ are perpendicular to $\overline{BC}$ and $\overline{AC}$ respectively, $\therefore \overrightarrow{OM} \bullet \overrightarrow{BC} = \frac{1}{2}(\mathbf{c} + \mathbf{b}) \bullet (\mathbf{c} - \mathbf{b}) = 0$ and $\overrightarrow{ON} \bullet \overrightarrow{AC} = \frac{1}{2}(\mathbf{c} + \mathbf{a}).(\mathbf{c} - \mathbf{a}) = 0$. Hence $\lvert\mathbf{c}\rvert^2 - \lvert\mathbf{b}\rvert^2 = 0$ and $\lvert\mathbf{c}\rvert^2 - \lvert\mathbf{a}\rvert^2 = 0$, $\therefore \lvert\mathbf{a}\rvert = \lvert\mathbf{b}\rvert = \lvert\mathbf{c}\rvert$.
Specific behaviours
✓ solves $\overrightarrow{OM} \bullet \overrightarrow{BC} = 0$ and thereby concludes that $\lvert\mathbf{b}\rvert = \lvert\mathbf{c}\rvert$. ✓ similarly uses $\overrightarrow{ON} \bullet \overrightarrow{AC} = 0$ and thereby concludes that $\lvert\mathbf{a}\rvert = \lvert\mathbf{c}\rvert$ so deduces that the three vectors all have the same magnitudes

(c) If P is the midpoint of side AB, show that $\overrightarrow{OP}$ is perpendicular to $\overrightarrow{AB}$. (3 marks)

Solution
$\overrightarrow{OP} = \frac{1}{2}$**(a + b)**. $\overrightarrow{OP} \bullet \overrightarrow{BA} = \frac{1}{2}$**(a + b)•(a – b)** = $\frac{1}{2}$ [|**a**|^2 – |**b**|^2] = 0, $\therefore \overrightarrow{OP}$ is perpendicular to $\overrightarrow{BA}$
Specific behaviours
✓ expresses $\overrightarrow{OP}$ in terms of **a** and **b** ✓ expresses dot product of $\overrightarrow{OP}$ and $\overrightarrow{BA}$ in terms of **a** and **b** ✓ shows that the above dot product equals zero

Question 5 **(11 marks)**

A mining drill moves through a cross-section of rock such that its coordinates x and y satisfy:

$$\frac{dx}{dt} = -2\sin t, \quad \frac{dy}{dt} = -6\sin 2t$$

where x and y are measured in metres and t is the time expressed in minutes.
Initially $x = 2$ and $y = 3$.

(a) Determine $\frac{dy}{dx}$ when $t = \frac{\pi}{6}$. (2 marks)

Solution
Now $\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{-6\sin 2t}{-2\sin t}$.   When $t = \frac{\pi}{6}$ then $\frac{dy}{dx} = \frac{3\sin\left(\frac{\pi}{3}\right)}{\sin\left(\frac{\pi}{6}\right)} = \frac{3\left(\frac{\sqrt{3}}{2}\right)}{\left(\frac{1}{2}\right)} = 3\sqrt{3}$.
Specific behaviours
✓ uses chain rule correctly   ✓ evaluates derivative at the specified time

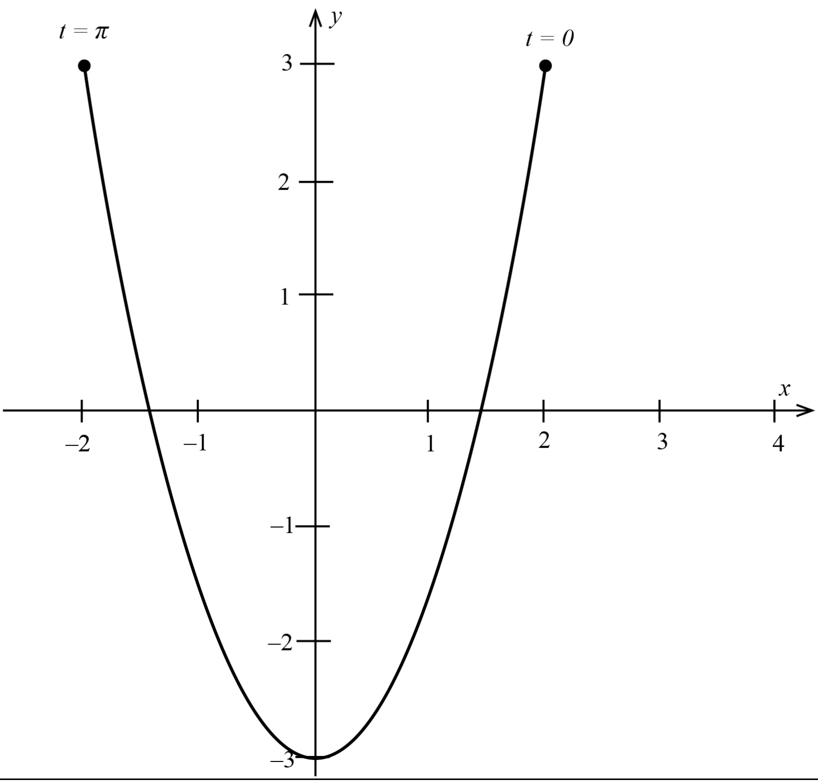
(b) Determine the position of the drill when $t = \pi$. (3 marks)

Solution
Integrating the differential equations gives $x = 2\cos t$ and $y = 3\cos 2t$ on applying the initial conditions.   When $t = \pi$ it follows that $x = -2$ and $y = 3$.
Specific behaviours
✓ determines expression for x in terms of t ✓ determines expression for y in terms of t ✓ determines x and y at the required time

(c) Determine the Cartesian equation of the path of the drill. (3 marks)

Solution
$y = 3\cos 2t = 3(2\cos^2 t - 1)$ $= 3\left(2\left(\frac{x}{2}\right)^2 - 1\right)$ $= \frac{3}{2}x^2 - 3$
Specific behaviours
✓ uses trig identity for $\cos 2t$ ✓ replaces $\cos t$ with $\frac{x}{2}$ ✓ expresses y in terms of x correctly

(d) Sketch the path of the drill. Label clearly the points on the path corresponding to the two times $t = 0$ and $t = \pi$. (3 marks)

Solution

Specific behaviours
✓ gives a neat sketch of the parabola ✓ clearly labels points (–2, 3) when $t = \pi$ and (2, 3) when $t = 0$ ✓ indicates that the parabola does not extend outside the interval $-2 \le x \le 2$

Question 6 **(7 marks)**

Describe geometrically the set of points **r** $= (x, y, z)$ that satisfy each of the following vector equations in which **a** denotes a non-zero constant vector.

(a) $|\mathbf{r} - \mathbf{a}| = 3$ (2 marks)

Solution
Sphere with centre **a** and radius of 3 units
Specific behaviours
✓ recognises that the shape is a sphere ✓ correctly identifies the centre and radius of the sphere

(b) $(\mathbf{r} - \mathbf{a}) \bullet \mathbf{a} = 0$ (2 marks)

Solution
Plane with a normal parallel to **a** and passing through the point with position vector **a**.
Specific behaviours
✓ identifies the locus as a plane ✓ characterises the plane completely; i.e. specifies a point on the plane and the direction of the normal

(c) $(\mathbf{r} - \mathbf{a}) \bullet \mathbf{r} = 0$ (3 marks)

Solution
A sphere with the origin and point with position vector **a** opposite ends of a diameter. Equivalently, the position vector of the centre of the sphere is $\frac{\mathbf{a}}{2}$ and its radius is $\frac{\lvert\mathbf{a}\rvert}{2}$.
Specific behaviours
✓ notes that the points lie on a sphere ✓✓ characterises the location of the sphere properly; e.g. specifies centre and radius or identifies a diameter of the sphere

Question 7 **(5 marks)**

(a) A square matrix is said to be singular if it has no inverse.

Show that the matrix $X = \begin{bmatrix} \cos 72° & \sin 18° \\ \sin 72° & \cos 18° \end{bmatrix}$ is singular. (2 marks)

Solution
$\det X = \cos 72^\circ \cos 18^\circ - \sin 18^\circ \sin 72^\circ$ $= \cos\left(72^\circ + 18^\circ\right)$ $= \cos 90^\circ = 0$
Specific behaviours
✓ writes down correct expression for the determinant ✓ uses trigonometric identity to show that determinant is zero

(b) Let A be any square matrix such that $A^3 = 0$.
Prove by contradiction that A must be singular. (3 marks)

Solution
Suppose that A has an inverse A^{-1} such that $AA^{-1} = I$ Multiply each side of $A^3 = 0$ by $A^{-1}A^3 = A^{-1}0$ $A^2 = 0$ multiply again $A^{-1}A^2 = A^{-1}0$ $A = 0$ $\therefore AA^{-1} = 0$ contradiction $\therefore A$ does not have an inverse
Specific behaviours
✓ supposes that A has an inverse ✓ deduces that $A^3 = 0$ or $A = 0$ ✓ concludes that A cannot have an inverse

End of questions

Published by the School Curriculum and Standards Authority of Western Australia
27 Walters Drive
OSBORNE PARK WA 6017